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Integrable relativistic Toda type lattice hierarchies, associated coupling systems and the Darboux transformation

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Abstract
Starting from a discrete isospectral problem, integrable positive and negative relativistic Toda type lattice hierarchies are derived. The two lattice hierarchies are proven to have discrete zero-curvature representations associated with a discrete spectral problem, and the positive and negative lattice hierarchies correspond to positive and negative power expansions of Lax operators with respect to the spectral parameter, respectively. The integrable positive and negative coupling systems of the resulting hierarchies are constructed through enlarging Lax pairs. In addition, with the help of gauge transformations of spectral problems, a Darboux transformation is established for the relativistic Toda type lattice. As an application, an exact solution is explicitly presented.

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1. Introduction

Nonlinear integrable lattice soliton equations are well known to be used for modelling physical phenomena such as particle vibrations in lattice, currents in electrical networks, pulses in biological chains, etc. The study of integrable systems of lattice versions, therefore, has aroused increasing interest in the last few years. Since the original work of Fermi, Pasta and Ulam in the 1950s [1], several physically important nonlinear integrable lattice soliton equations have been obtained and systematically discussed [2–11]. The discrete matrix spectral problem, i.e., the Lax pair of lattice soliton equation plays a crucial role in the treatment of lattice soliton theory. From the associated Lax pairs, many new lattice soliton systems and the interrelated nice properties, such as the nonlinearization, the master symmetries, the infinite conservation laws, the Darboux transformation and so on, can usually be investigated conveniently.

In addition, there is another natural problem, that is how to extend the known lattice systems to larger and complicated models from points of view of both potentials and dimensions. Recently, the study of integrable couplings of soliton equations has attracted much attention. The study of integrable couplings of soliton equations originates from the investigations into the symmetry problems and associated centreless Virasoro algebras [12]. It can be viewed as an approach to obtain new larger integrable hierarchies in the Lax sense and shows one plenty of integrable structures that the multiplicity of integrable systems brings.

In the field of nonlinear lattice theory, moreover, to obtain exact solutions for the lattice equations is always one of the most fundamental and significant topics. While there has been considerable work done on finding exact solutions [13–15] to lattice equations, as far as we could verify, the Darboux transformation technique is still an effective method which can always be used to explicitly present the exact solutions for lattice soliton equations and has also been widely used by many authors [10, 11, 16–21, and references therein].

The present paper is devoted to introducing a discrete isospectral problem

$$E\varphi_n(\lambda) = U_n(u_n,\lambda)\varphi_n(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda r_n & \lambda + s_n \end{pmatrix}\varphi_n(\lambda),\ \varphi_n(\lambda) = \begin{pmatrix} \varphi_n^1(\lambda) \\ \varphi_n^2(\lambda) \end{pmatrix}, \tag{1}$$

where the shift operator E, the inverse of E and the difference operator D are defined as $Ef_n = f(n+1) = f_{n+1}$, $E^{-1}f_n = f_{n-1}$, $Df_n = f_{n+1} - f_n = (E-1)f_n$, $n \in Z$, $u_n = (r_n, s_n)^T$ is a two-component potential function vector defined over $R \times Z$ and rapidly vanishes if $|n| \to \infty$ and λ is the spectral parameter with $\lambda_t = 0$. By means of constructing a proper continuous time evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_n(\lambda) = \Gamma_n^{\{m\}}(u_n,\lambda)\varphi_n(\lambda), \tag{2}$$

where $\Gamma_n^{\{m\}}(u_n,\lambda)$ is a suitable 2×2 matrix, and using the discrete zero-curvature equation

$$\frac{\mathrm{d}}{\mathrm{d}t}U_n - \left(E\Gamma_n^{\{m\}}\right)U_n + U_n\Gamma_n^{\{m\}} = 0, \tag{3}$$

a pair of positive and negative hierarchies of integrable nonlinear lattice models is derived [4, 8]. As the typical cases, the first two nonlinear lattice equations are given as follows, respectively,

$$r_{n_t} = r_n(s_{n-1} - s_n) + r_n(r_{n-1} - r_{n+1}), \qquad s_{n_t} = s_n(r_n - r_{n+1}), \tag{4a}$$

and

$$r_{n_t} = \frac{r_n}{s_{n-1}} - \frac{r_n}{s_n}, \qquad s_{n_t} = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_{n-1}}. \tag{4b}$$

It is shown that the resulting lattice equations correspond to positive and negative power expansions of Lax operators with respect to the spectral parameter, respectively, and each equation in resulting hierarchies is Liouville integrable. In virtue of [7], the lattice equation (4*a*) can be called relativistic Toda type lattice, and equation (4*b*) as its negative flow. Through enlarging associated Lax pair [22–25], the integrable positive and negative coupling systems of relativistic Toda type lattice hierarchy are constructed. Furthermore, by virtue of gauge transformation, a Darboux transformation for spectral problem (1) is constructed. As application, exact solutions of positive relativistic Toda type lattice equations (4*a*) are given.

2. Integrable positive and negative relativistic Toda type lattices

In this section, we would like to derive the positive and negative relativistic Toda type lattice hierarchies based on the spectral problem (1). We first solve the stationary discrete zero-curvature equation [3]

$$(EV_n)U_n - U_nV_n = 0, \tag{5}$$

where

$$V_n = \sum_{m\geqslant 0} \begin{pmatrix} a_n^{(m)}\lambda^{-m} & b_n^{(m)}\lambda^{-m} \\ c_n^{(m)}\lambda^{-m+1} & -a_n^{(m)}\lambda^{-m} \end{pmatrix}.$$

Then, equation (5) leads to the initial relations

$$a_n^{(0)} = -\tfrac{1}{2}, \qquad b_n^{(0)} = c_n^{(0)} = 0,$$

and the recursion relations

$$\begin{aligned} & r_n b_{n+1}^{(m)} = c_n^{(m)}, \\ & b_{n+1}^{(m+1)} + a_{n+1}^{(m)} + a_n^{(m)} + s_n b_{n+1}^{(m)} = 0, \\ & c_n^{(m+1)} + r_n\left(a_{n+1}^{(m)} + a_n^{(m)}\right) + s_n c_n^{(m)} = 0, \\ & a_{n+1}^{(m+1)} - a_n^{(m+1)} = c_{n+1}^{(m+1)} - r_n b_n^{(m+1)} - s_n\left(a_{n+1}^{(m)} - a_n^{(m)}\right), \end{aligned} \qquad m \geqslant 0, \tag{6}$$

which are all difference polynomials in u_n with respect to the lattice variable n. Under the initial-value conditions of selecting zero constants for the inverse operation of the difference operator D in computing $a_n^{(m)}$, $m \geqslant 1$, the recursion relations (6) uniquely determine $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}$, $m \geqslant 1$ and the first few quantities are given by

$$\begin{aligned} & a_n^{(1)} = r_n, b_n^{(1)} = 1, c_n^{(1)} = r_n \\ & b_{n+1}^{(2)} = -r_n - r_{n+1} - s_n, c_n^{(2)} = -r_n(r_n + r_{n+1}) - r_n s_n, \\ & a_n^{(2)} = -r_n s_n - r_n s_{n-1} - r_n^2 - r_n r_{n-1} - r_n r_{n+1} \ldots . \end{aligned}$$

Moreover, from (5), we know [3, 8] that $(E-1)\,\mathrm{tr}\left(V_n^k\right) = 0$ for all $k \geqslant 1$. In particular, we have $\mathrm{tr}\left(V_n^2\right) = 2\left(a_n^2 + \lambda b_n c_n\right)$ is a constant, and let us say γ. Then, we obtain a recursion relation for $a_n^{(m)}$

$$a_n^{(m+1)} = \sum_{i=1}^{m} a_n^{(i)} a_n^{(m-i+1)} + \sum_{i=1}^{m+1} b_n^{(i)} c_n^{(m-i+1)} - \frac{1}{2}\gamma, \qquad m \geqslant 1. \tag{7}$$

This, together with the first two equations in (6), implies that all lattice functions $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}$, $m \geqslant 1$, are local and they are just difference polynomials in r_n and s_n.

Now, we set

$$(\lambda^m V_n)_+ = \sum_{i=0}^{m} \begin{pmatrix} a_n^{(i)}\lambda^{m-i} & b_n^{(i)}\lambda^{m-i} \\ c_n^{(i)}\lambda^{m-i+1} & -a_n^{(i)}\lambda^{m-i} \end{pmatrix}.$$

It is not difficult to find

$$(E(\lambda^m V_n)_+)U_n - U_n(\lambda^m V_n)_+ = \begin{pmatrix} 0 & -b_{n+1}^{(m+1)} \\ \lambda c_n^{(m+1)} & s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \end{pmatrix}.$$

So we introduce the modification as follows:

$$\Delta_n = \begin{pmatrix} b_n^{(m+1)} & 0 \\ 0 & 0 \end{pmatrix},$$

and define [3]

$$V_n^{\{+m\}} = (\lambda^m V_n)_+ + \Delta_n, \qquad m \geqslant 0. \tag{8}$$

Through a direct calculation, we have

$$\left(EV_n^{\{+m\}}\right)U_n - U_n V_n^{\{+m\}} = \begin{pmatrix} 0 & 0 \\ \lambda\left(c_n^{(m+1)} - rb_n^{(m+1)}\right) & s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \end{pmatrix},$$

which is consistent with $U_{n_{t_m}}$. Then, we introduce the following auxiliary spectral problem:

$$\varphi_{n_{t_m}} = V_n^{\{+m\}}\varphi_n, \qquad m \geqslant 0, \tag{9}$$

the compatibility conditions of (1) and (9), i.e., the discrete zero-curvature equations

$$U_{n_{t_m}} = \left(EV_n^{\{+m\}}\right)U_n - U_n V_n^{\{+m\}}, \tag{10}$$

give rise to the following positive hierarchy of lattice soliton equations:

$$\begin{cases} r_{n_{t_m}} = c_n^{(m+1)}, \\ s_{n_{t_m}} = s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right), \end{cases} \qquad m \geqslant 0. \tag{11}$$

When $m = 1$, the resulting lattice system (11) reduces to the positive relativistic Toda type lattice equation (4*a*). The temporal evolution laws of equation (4*a*) are as follows:

$$(\varphi_n(\lambda))_{t_1} = V_n^{\{+1\}}(u,\lambda)\varphi_n(\lambda), \qquad V_n^{\{+1\}} = \begin{pmatrix} -\frac{1}{2}\lambda - r_{n-1} - s_{n-1} & 1 \\ \lambda r_n & \frac{1}{2}\lambda - r_n \end{pmatrix}. \tag{12}$$

So, one can call the lattice system (11) as relativistic Toda type lattice hierarchy.

In what follows, we would like to construct the Hamiltonian structure for the lattice system (11). To this end, we apply the trace identity [3]

$$\frac{\delta}{\delta u_n}\sum_{k\in Z}\left\langle \Gamma_n, \frac{\partial U_n}{\partial \lambda}\right\rangle(k) = \left(\lambda^{-\varepsilon}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\varepsilon}\right)\left\langle \Gamma_n, \frac{\partial U_n}{\partial u_{n_i}}\right\rangle, \qquad i = 1, 2,$$

where $\Gamma_n = V_n U_n^{-1}$ and $\langle A, B\rangle = \mathrm{tr}(AB)$, where A, B are some order square matrix. Through direct calculations, system (11) has a bi-Hamiltonian structure [4]

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J_1 \frac{\delta \widetilde{H}_{m+1}}{\delta u_n} = J_1 \begin{pmatrix} \frac{a_n^{(m+1)}}{r_n} \\ \frac{c_n^{(m+1)}}{r_n} \end{pmatrix} = M \begin{pmatrix} \frac{a_n^{(m)}}{r_n} \\ \frac{c_n^{(m)}}{r_n} \end{pmatrix}, \qquad m \geqslant 0, \tag{13}$$

where variational derivatives are defined by

$$\frac{\delta\widetilde{H}}{\delta u_n} = \left(\frac{\delta\widetilde{H}}{\delta r_n}, \frac{\delta\widetilde{H}}{\delta s_n}\right)^T, \qquad \frac{\delta\widetilde{H}}{\delta r_n} = \sum_{m\in Z} E^{-m}\left(\frac{\partial\widetilde{H}}{\partial r_{n+m}}\right), \qquad \frac{\delta\widetilde{H}}{\delta s_n} = \sum_{m\in Z} E^{-m}\left(\frac{\partial\widetilde{H}}{\partial s_{n+m}}\right),$$

the Hamiltonian operators J_1, M and the Hamiltonian functionals $\widetilde{H}_m$ are given by

$$J_1 = \begin{pmatrix} 0 & r_n(1-E^{-1}) \\ (E-1)r_n & r_nE^{-1} - Er_n \end{pmatrix}, \qquad M = \begin{pmatrix} r_n(E^{-1}-E)r_n & r_n(1-E^{-1})s_n \\ s_n(E-1)r_n & 0 \end{pmatrix}, \tag{14}$$

$$\widetilde{H}_0 = -\frac{1}{2}\sum_{k\in Z}(\ln r_n)(k), \qquad \widetilde{H}_m = \sum_{k\in Z}\left(-\frac{r_n a_n^{(m)} + c_n^{(m)}}{m r_n}\right)(k), \qquad m \geqslant 1.$$

Now, let us note

$$\frac{\delta\widetilde{H}_m}{\delta u_n} = \Phi\frac{\delta\widetilde{H}_{m-1}}{\delta u_n}, \qquad \Phi = J_1^{-1}M = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{15}$$

With the help of recursion relations (6), we have

$$\Phi_{11} = -(1+E)r_n - \frac{1}{r_n}(E-1)^{-1}[r_n(E-E^{-1})r_n - s_n(1-E)r_n],$$
$$\Phi_{12} = -s_n - \frac{1}{r_n}(E-1)^{-1}r_n(1-E^{-1})s_n,$$
$$\Phi_{21} = -(1+E)r_n, \qquad \Phi_{22} = -s_n.$$

Therefore, starting from the discrete spectral problem (1), the positive lattice hierarchy (11) is derived. It is easy to verify that the positive hierarchy has the discrete zero-curvature representation (10). And, every soliton equation in (11) or the discrete Hamiltonian system (13) is a discrete Liouville integrable system [3]. Further, it is easy to verify that the operator Φ is invertible, and its inverse operator can be given by

$$\Psi := \Phi^{-1} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}, \tag{16}$$

with

$$\Psi_{11} = \frac{1}{r_n}(E-1)^{-1}\frac{1}{s_n}(1-E)r_n,$$
$$\Psi_{12} = \frac{1}{r_n}(E-1)^{-1}\frac{1}{s_n}(Er_n - r_nE^{-1}),$$
$$\Psi_{21} = -\frac{1}{s_n}(1+E)(E-1)^{-1}\frac{r_n}{s_n}(1-E)r_n,$$
$$\Psi_{22} = -\frac{1}{s_n}(1+E)(E-1)^{-1}\frac{1}{s_n}(Er_n - r_nE^{-1}) - \frac{1}{s_n}.$$

Now, we would like to briefly derive the negative hierarchy of lattice soliton equations based on the spectral problem (1). To this end, we introduce the auxiliary spectral matrix

$$W_n = \sum_{m=0}^{\infty} \begin{pmatrix} A_n^{(m)}\lambda^m & B_n^{(m)}\lambda^m \\ C_n^{(m)}\lambda^{m+1} & -A_n^{(m)}\lambda^m \end{pmatrix}.$$

Hence, the stationary discrete zero-curvature equation,

$$(EW_n)U_n - U_nW_n = 0,$$

implies that

$$A_n^{(0)} = -\frac{1}{2}, \qquad B_{n+1}^{(0)} = \frac{1}{s_n}, \qquad C_n^{(0)} = \frac{r_n}{s_n}, \tag{17}$$

$$\left.\begin{aligned} & r_nB_{n+1}^{(m)} = C_n^{(m)}, \\ & A_{n+1}^{(m+1)} + A_n^{(m+1)} + B_{n+1}^{(m)} + s_nB_{n+1}^{(m+1)} = 0, \\ & r_n\left(A_{n+1}^{(m+1)} + A_n^{(m+1)}\right) + C_n^{(m)} + s_nC_n^{(m+1)} = 0, \\ & s_n\left(A_{n+1}^{(m+1)} - A_n^{(m+1)}\right) = C_{n+1}^{(m)} - r_nB_n^{(m)} + A_n^{(m)} - A_{n+1}^{(m)}, \end{aligned}\right\} \quad m \geqslant 0, \tag{18}$$

which uniquely define $A_n^{(m)}, B_n^{(m)}, C_n^{(m)}, m \geqslant 1$ and the first few quantities are as follows:

$$A_n^{(1)} = \frac{r_n}{s_{n-1}s_n},$$
$$s_nB_{n+1}^{(1)} - \left(\frac{r_n}{s_{n-1}s_n} + \frac{r_{n+1}}{s_ns_{n+1}}\right) - \frac{1}{s_n},$$
$$s_nC_n^{(1)} = -r_n\left(\frac{r_n}{s_{n-1}s_n} + \frac{r_{n+1}}{s_ns_{n+1}}\right) - \frac{r_n}{s_n}, \ldots.$$

Similarly, all lattice functions $A_n^{(m)}$, $B_n^{(m)}$, $C_n^{(m)}$, $m \geqslant 1$, defined by equations (18) are local and they are just difference polynomials in r_n and s_n. Now, we set

$$W_n^{\{-m\}} = \sum_{i=0}^{m} \begin{pmatrix} A_n^{(i)}\lambda^{-m+i-1} - B_n^{(m)} & B_n^{(i)}\lambda^{-m+i-1} \\ C_n^{(i)}\lambda^{-m+i} & -A_n^{(i)}\lambda^{-m+i-1} \end{pmatrix}, \qquad m \geqslant 0. \tag{19}$$

Then, the discrete zero-curvature equations

$$U_{n_{t_m}} = \left(EW_n^{\{-m\}}\right)U_n - U_n W_n^{\{-m\}}, \tag{20}$$

lead to the following negative hierarchy of lattice soliton equations:

$$\begin{cases} r_{n_{t_m}} = r_n B_n^{(m)} - C_n^{(m)}, \\ s_{n_{t_m}} = s_n\left(A_{n+1}^{(m+1)} - A_n^{(m+1)}\right), \end{cases} \qquad m \geqslant 0, \tag{21}$$

where the first nonlinear lattice system, when $m = 0$, is given by (4*b*) coming from the negative lattice hierarchy (21). The temporal evolution laws of equation (4*b*) can be shown as

$$(\varphi_n(\lambda))_{t_0} = W_n^{\{0\}}(u,\lambda)\varphi_n(\lambda), \qquad W_n^{\{0\}} = \begin{pmatrix} -\frac{1}{2\lambda} - \frac{1}{s_{n-1}} & \frac{1}{s_{n-1}\lambda} \\ \frac{r_n}{s_n} & \frac{1}{2\lambda} \end{pmatrix}. \tag{22}$$

By using equations (18), the discrete system (21) can be rewritten in the Hamiltonian form

$$u_{t_m} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J_2 \frac{\delta \widetilde{G}_m}{\delta u_n} = J_2 \begin{pmatrix} \frac{A_n^{(m)}}{r_n} \\ \frac{C_n^{(m)}}{r_n} \end{pmatrix} = J_2 \Psi^m \begin{pmatrix} -\frac{1}{2r_n} \\ \frac{1}{s_n} \end{pmatrix}, \qquad m \geqslant 1, \tag{23}$$

where

$$\widetilde{G}_m = \sum_{k\in Z}\left(\frac{r_n A_n^{(m)} + C_n^{(m-1)}}{m r_n}\right)(k),\ m \geqslant 1, \qquad \widetilde{G}_0 = \sum_{k\in Z}\left(\ln(r_n)^{-\frac{1}{2}} s_n\right)(k),$$

where $J_2 = -J_1$ is obviously a Hamiltonian operator and Ψ is defined by (16). Moreover, we obtain

$$N = J_2\Psi = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix},$$

with

$$N_{11} = -r_n(1 - E^{-1})\frac{1}{s_n}(1 + E)(E - 1)^{-1}\frac{1}{s_n}(1 - E)r_n,$$

$$N_{12} = r_n(1 - E^{-1})\frac{1}{s_n}\left[1 - (1 + E)(E - 1)^{-1}\frac{1}{s_n}(Er_n - r_n E^{-1})\right],$$

$$N_{21} = \left[1 - (r_n E^{-1} - Er_n)\frac{1}{s_n}(1 + E)(E - 1)^{-1}\right]\frac{1}{s_n}(1 - E)r_n,$$

$$N_{22} = (r_n E^{-1} - Er_n)\frac{1}{s_n}(1 + E)(1 - E)^{-1}\frac{1}{s_n}(Er_n - r_n E^{-1}) + \frac{1}{s_n}(Er_n - r_n E^{-1}) - (r_n E^{-1} - Er_n)\frac{1}{s_n}.$$

It is not difficult to verify that N is a skew-symmetry operator. Hence, similarly, from the discrete spectral problem (1), the negative lattice hierarchy (21) is derived. It is shown that the negative hierarchy has the discrete zero-curvature representation (20). And, every soliton equation in (21) or the discrete Hamiltonian system (23) possesses the Liouville integrability [3].

It is known that if J is a Hamiltonian operator, then

$$\left[J\frac{\delta\widetilde{H}_1}{\delta u_n}, J\frac{\delta\widetilde{H}_2}{\delta u_n}\right] = J\frac{\delta\{\widetilde{H}_1, \widetilde{H}_2\}_J}{\delta u_n},$$

where the commutator is defined by

$$[X, \Theta] := \frac{\partial}{\partial\varepsilon}(X(u_n + \varepsilon\Theta) - \Theta(u_n + \varepsilon X))|_{\varepsilon=0}.$$

Thus, for the two hierarchies of lattice equations, we have

$$[X_m, X_l] = \left[J_1\frac{\delta\widetilde{H}_m}{\delta u_n}, J_1\frac{\delta\widetilde{H}_l}{\delta u_n}\right] = J_1\frac{\delta\{\widetilde{H}_m, \widetilde{H}_l\}_{J_1}}{\delta u_n}, \qquad m, l \geqslant 0,$$

and

$$[\Theta_m, \Theta_l] = \left[J_2\frac{\delta\widetilde{G}_m}{\delta u_n}, J_2\frac{\delta\widetilde{G}_l}{\delta u_n}\right] = J_2\frac{\delta\{\widetilde{G}_m, \widetilde{G}_l\}_{J_2}}{\delta u_n}, \qquad m, l \geqslant 0,$$

in which the commutativity of the Hamiltonian functionals is a consequence of the recursion relation

$$\frac{\delta\widetilde{H}_{m+1}}{\delta u_n} = \Phi\frac{\delta\widetilde{H}_m}{\delta u_n}, \qquad \frac{\delta\widetilde{G}_{m+1}}{\delta u_n} = \Psi\frac{\delta\widetilde{G}_m}{\delta u_n}, \qquad m \geqslant 0.$$

Therefore, $\{X_m\}_{m=0}^{\infty}$ and $\{\widetilde{H}_m\}_{m=0}^{\infty}$ are infinitely many commuting symmetries and infinitely many commuting conserved functionals of the positive hierarchy (11), and $\{\Theta_m\}_{m=0}^{\infty}$ and $\{\widetilde{G}_m\}_{m=0}^{\infty}$ are infinitely many commuting symmetries and infinitely many commuting conserved functionals of the negative hierarchy (21).

3. Integrable positive and negative coupling systems for the lattice hierarchy (11)

Mathematically, for a given integrable lattice hierarchy of evolution type

$$u_t = K(u) = K(\ldots, E^{-1}u, u, Eu, \ldots), \tag{24}$$

we actually want to construct a new bigger, triangular system

$$u_t = K(u), \qquad v_t = S(u, v), \tag{25}$$

as an integrable coupling of the former system (24) if it is still integrable. The vector-value function S should satisfy the non-triviality condition $\frac{\partial S}{\partial[u]} \neq 0$, where $[u]$ denotes a vector function consisting of all $(u, v, Eu, Ev, E^{-1}u, E^{-1}v, \ldots)$. How to construct integrable couplings for a given system is a primary problem. Recently, Ma [22] proposed a method to obtain integrable coupling systems, through enlarging associated spectral problems, for known integrable equations, by means of which the integrable couplings of AKNS hierarchy and multi-component version of AKNS equations [23] are successfully investigated, respectively. In [24], authors have first extended the procedure to the semi-discrete systems, where the integrable coupling of generalized Toda lattice is discussed. More recently, an interesting conclusion, proposed by Sakovich, shows that this method also gives some non-integrable systems as well [26].

In this section, we would like to construct the positive and negative integrable coupling systems for the relativistic Toda type lattice hierarchy (11) through enlarging spectral problem (1). To this end, we consider the enlarged spectral problem

$$E\bar{\varphi}_n(\lambda) = \bar{U}_n(\bar{u}_n, \lambda)\bar{\varphi}_n(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda r_n & \lambda + s_n & w_n\lambda \\ 0 & 0 & 0 \end{pmatrix} \bar{\varphi}_n(\lambda), \tag{26}$$

where $\bar{u}_n = (r_n, s_n, w_n)^T$ and $\bar{\varphi}_n(\lambda) = \left(\bar{\varphi}_n^1(\lambda), \bar{\varphi}_n^2(\lambda), \bar{\varphi}_n^3(\lambda)\right)^T$.

Let

$$\bar{\varphi}_{n_{t_m}}(\lambda) = \bar{V}_n^{\{+m\}}(\bar{u}_n, \lambda)\bar{\varphi}(\lambda), \tag{27}$$

with

$$\bar{V}_n^{\{+m\}} = \sum_{i=0}^{m} \begin{pmatrix} a_n^{(i)}\lambda^{m-i} + b_n^{(m+1)} & b_n^{(i)}\lambda^{m-i} & d_n^{(i)}\lambda^{m-i} \\ c_n^{(i)}\lambda^{m-i+1} & -a_n^{(i)}\lambda^{m-i} & f_n^{(i)}\lambda^{m-i+1} \\ 0 & 0 & 0 \end{pmatrix}, \qquad m \geqslant 0. \tag{28}$$

Then $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}, d_n^{(m)}, f_n^{(m)}, m \geqslant 0$ satisfy the following initial relations:

$$a_n^{(0)} = -\frac{1}{2}, \qquad b_n^{(0)} = c_n^{(0)} = f_n^{(0)} = 0, \qquad d_n^{(0)} = -\frac{w_n}{2r_n},$$

and the recursion relations

$$\begin{aligned} & r_n b_{n+1}^{(m)} = c_n^{(m)}, \\ & b_{n+1}^{(m+1)} + a_{n+1}^{(m)} + a_n^{(m)} + s_n b_{n+1}^{(m)} = 0, \\ & c_n^{(m+1)} + r_n\left(a_{n+1}^{(m)} + a_n^{(m)}\right) + s_n c_n^{(m)} = 0, \\ & a_{n+1}^{(m+1)} - a_n^{(m+1)} = c_{n+1}^{(m+1)} - r_n b_n^{(m+1)} - s_n\left(a_{n+1}^{(m)} - a_n^{(m)}\right), \\ & w_n b_{n+1}^{(m+1)} = f_n^{(m+1)}, \\ & w_n a_{n+1}^{(m)} + r_n d_n^{(m)} + s_n f_n^{(m)} + f_n^{(m+1)} = 0. \end{aligned} \qquad m \geqslant 0, \tag{29}$$

From the fifth and second equalities in (29), we have

$$f_n^{(m+1)} = -w_n\left(a_{n+1}^{(m)} + a_n^{(m)}\right) - s_n w_n b_{n+1}^{(m)},$$

which, noting the sixth equality in (29), gives

$$r_n d_n^{(m)} = w_n\left(a_n^{(m)} + s_n b_{n+1}^{(m)}\right) - s_n f_n^{(m)},$$

which guarantees the solvability of the quantities $d_n^{(m)}, m \geqslant 0$. The enlarged discrete zero-curvature equations

$$\bar{U}_{n_{t_m}} = \left(E\bar{V}_n^{\{+m\}}\right)\bar{U}_n - \bar{U}_n\bar{V}_n^{\{+m\}}, \qquad m \geqslant 0,$$

give rise to the following positive hierarchy of lattice soliton equations

$$\bar{u}_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ w_n \end{pmatrix}_{t_m} = \begin{pmatrix} c_n^{(m+1)} \\ s_n\left(a_n^{(m)} - a_{n+1}^{(m)}\right) \\ f_n^{(m+1)} \end{pmatrix} = \bar{J}_1 \begin{pmatrix} \frac{a_n^{(m+1)}}{r_n} \\ \frac{c_n^{(m+1)}}{r_n} \\ b_{n+1}^{(m+1)} \end{pmatrix} = \bar{J}_1\bar{\Phi} \begin{pmatrix} \frac{a_n^{(m)}}{r_n} \\ \frac{c_n^{(m)}}{r_n} \\ b_{n+1}^{(m)} \end{pmatrix}, \quad m \geqslant 0. \tag{30}$$

Here

$$\bar{J}_1 = \begin{pmatrix} J_1 & 0 \\ 0 & w_n \end{pmatrix},$$

where J_1 is defined by (14) and

$$\bar{\Phi} = \begin{pmatrix} \Phi & 0 \\ \bar{\Phi}_{21} & -s_n \end{pmatrix}, \qquad \bar{\Phi}_{21} = (-(1+E)r_n, 0),$$

where Φ is defined by equation (15).

According to the definition of integrable coupling [12], equation (30) is a kind of integrable coupling of equation (11). When $m = 1$, the resulting lattice system (30) reduces to the first nonlinear lattice equation

$$\begin{cases} r_{n_t} = r_n(s_{n-1} - s_n) + r_n(r_{n-1} - r_{n+1}), \\ s_{n_t} = s_n(r_n - r_{n+1}), \\ w_{n_t} = -w_n(r_{n+1} + r_n) - w_n s_n, \end{cases} \tag{31}$$

which is the integrable coupling system of lattice equation (4*a*) and will be equation (4*a*) when we set $w_n = 0$. It is easy to verify, noting the establishment of the enlarged temporal matrix, that the coupling system (30) corresponds to the positive power expansion of Lax operators with respect to the spectral parameter. It, therefore, can be called integrable positive coupling system of hierarchy (11). And equation (31) is the integrable positive coupling of the lattice equation (4*a*).

In what follows, the negative integrable coupling system will be deduced starting from the enlarged spectral problem (26). In order to do so, we set

$$\bar{\varphi}_{n_{t_m}}(\lambda) = \bar{W}_n^{\{-m\}}(\bar{u}_n, \lambda)\bar{\varphi}(\lambda), \tag{32}$$

with

$$\bar{W}_n^{\{-m\}} = \sum_{i=0}^{m} \begin{pmatrix} A_n^{(i)}\lambda^{-m+i-1} - B_n^{(m)} & B_n^{(i)}\lambda^{-m+i-1} & D_n^{(i)}\lambda^{-m+i-1} \\ C_n^{(i)}\lambda^{-m+i} & -A_n^{(i)}\lambda^{-m+i-1} & F_n^{(i)}\lambda^{-m+i} \\ 0 & 0 & 0 \end{pmatrix}, \qquad m \geqslant 0. \tag{33}$$

Then $A_n^{(m)}, B_n^{(m)}, C_n^{(m)}, D_n^{(m)}, F_n^{(m)}, m \geqslant 0$ satisfy the following initial relations

$$A_n^{(0)} = -\frac{1}{2}, \qquad B_{n+1}^{(0)} = \frac{1}{s_n}, \qquad C_n^{(0)} = \frac{r_n}{s_n}, \qquad D_n^{(0)} = -\frac{w_n}{2r_n}, \qquad F_n^{(0)} = \frac{w_n}{s_n},$$

and the recursion relations

$$\begin{aligned} & r_n B_{n+1}^{(m)} = C_n^{(m)}, \\ & A_{n+1}^{(m+1)} + A_n^{(m+1)} + B_{n+1}^{(m)} + s_n B_{n+1}^{(m+1)} = 0, \\ & r_n\big(A_{n+1}^{(m+1)} + A_n^{(m+1)}\big) + C_n^{(m)} + s_n C_n^{(m+1)} = 0, \\ & s_n\big(A_{n+1}^{(m+1)} - A_n^{(m+1)}\big) = C_{n+1}^{(m)} - r_n B_n^{(m)} + A_n^{(m)} - A_{n+1}^{(m)}, \\ & w_n B_{n+1}^{(m)} = F_n^{(m)}, \\ & w_n A_{n+1}^{(m)} + r_n D_n^{(m+1)} + s_n F_n^{(m+1)} + F_n^{(m)} = 0. \end{aligned} \qquad m \geqslant 0, \tag{34}$$

The enlarged discrete zero-curvature equations

$$\bar{U}_{n_{t_m}} = \big(E\bar{W}_n^{\{-m\}}\big)\bar{U}_n - \bar{U}_n\bar{W}_n^{\{-m\}}, \qquad m \geqslant 0,$$

give rise to the following negative hierarchy of lattice soliton equations:

$$\bar{u}_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \\ w_n \end{pmatrix}_{t_m} = \begin{pmatrix} r_n B_n^{(m)} - C_n^{(m)} \\ s_n\left(A_{n+1}^{(m+1)} - A_n^{(m+1)}\right) \\ -F_n^{(m)} \end{pmatrix} = \bar{J}_2 \begin{pmatrix} \frac{A_n^{(m)}}{r_n} \\ \frac{C_n^{(m)}}{r_n} \\ B_{n+1}^{(m)} \end{pmatrix} = \bar{J}_2\bar{\Psi}\begin{pmatrix} \frac{A_n^{(m-1)}}{r_n} \\ \frac{C_n^{(m-1)}}{r_n} \\ B_{n+1}^{(m-1)} \end{pmatrix}, \quad m \geqslant 0, \tag{35}$$

where $\bar{J}_2 = -\bar{J}_1$, and $\bar{J}_1$ is defined by (30) and

$$\bar{\Psi} = \begin{pmatrix} \Psi & 0 \\ \bar{\Psi}_{21} & \bar{\Psi}_{22} \end{pmatrix}, \qquad \bar{\Psi}_{21} = \big(\bar{\Psi}_{21}^1, \bar{\Psi}_{21}^2\big),$$

with

$$\bar{\Psi}_{21}^{1} = -\frac{1}{s_n}(1+E)(E-1)^{-1}\frac{1}{s_n}(1-E)r_n,$$
$$\bar{\Psi}_{21}^{2} = -\frac{1}{s_n}(1+E)(E-1)^{-1}\frac{1}{s_n}Er_n,$$
$$\bar{\Psi}_{22} = \frac{1}{s_n}(1+E)(E-1)^{-1}\frac{1}{s_n}r_nE^{-1} - \frac{1}{s_n}.$$

where Ψ is same as that of equation (16).

Hence, in light with the construction of the enlarged lattice system (35), it is a kind of negative integrable coupling of equation (11). When $m = 0$, the resulting lattice system (35) reduces to the first nonlinear lattice equation

$$\begin{cases} r_{n_t} = \frac{r_n}{s_{n-1}} - \frac{r_n}{s_n}, \\ s_{n_t} = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_{n-1}}, \\ w_{n_t} = -\frac{w_n}{s_n}, \end{cases} \tag{36}$$

the negative integrable coupling system of equation (4*a*). It is easy to verify that system (36) is an enlarged system of lattice equation (4*b*) and will be equation (4*b*) when we set $w_n = 0$.

4. A Darboux transformation and soliton solutions

In this section, a Darboux transformation (DT) for the relativistic Toda type lattice equation (4*a*) will be established. To this end, we first assume that there is a gauge transformation

$$\widetilde{\varphi}_n = T_n\varphi_n, \tag{37}$$

which can transform eigenvalue problem (1) and the auxiliary problem (12) into

$$\widetilde{\varphi}_{n+1} = \widetilde{U}_n\widetilde{\varphi}_n, \qquad \widetilde{\varphi}_{nt} = \widetilde{V}_n^{\{+1\}}\widetilde{\varphi}_n,$$

respectively, with

$$\widetilde{U}_n = T_{n+1}U_nT_n^{-1}, \qquad \widetilde{V}_n^{\{+1\}} = \left(T_{nt} + T_nV_n^{\{+1\}}\right)T_n^{-1}, \tag{38}$$

satisfying the fact that $\widetilde{U}_n, \widetilde{V}_n^{\{+1\}}$ and $U_n, V_n^{\{+1\}}$ have the same schedules, respectively.

Let $\phi_n = \left(\phi_n^1, \phi_n^2\right)^T$, $\psi_n = \left(\psi_n^1, \psi_n^2\right)^T$ be two background solutions of (1) and (12) and use (ϕ_n, ψ_n) to define a 2×2 matrix T_n by

$$T_n = \begin{pmatrix} (1-t_{12}(n))\lambda + t_{11}(n) & t_{12}(n) \\ \lambda t_{21}(n) & \lambda + t_{22}(n) \end{pmatrix}. \tag{39}$$

We are going to express the coefficients of T_n by ϕ_n and ψ_n. In order to do so, we assume that λ_1 and λ_2 are two solutions of $\det T_n = 0$. When $\lambda = \lambda_i$ $(i = 1, 2)$, we have

$$\widetilde{\varphi}_n = T_n\varphi_n$$
$$= \begin{pmatrix} (1-t_{12}(n))\lambda_i\phi_n^1 + \phi_n^1t_{11}(n) + \phi_n^2t_{12}(n) & (1-t_{12}(n))\lambda_i\psi_n^1 + \psi_n^1t_{11}(n) + \psi_n^2t_{12}(n) \\ \lambda_i\phi_n^1t_{21}(n) + \lambda_i\phi_n^2 + \phi_n^2t_{22}(n) & \lambda_i\psi_n^1t_{21}(n) + \lambda_i\psi_n^2 + \psi_n^2t_{22}(n) \end{pmatrix},$$

where two rows in matrix $\widetilde{\varphi}_n$ are linear correlated. Hence, there must be coefficients assumed γ_i $(i = 1, 2)$ such that

$$\begin{aligned} &(1-t_{12}(n))\lambda_i + t_{11}(n) + \alpha_i(n)t_{12}(n) = 0, \\ &\lambda t_{21}(n) + \alpha_i(n)(\lambda_i + t_{22}(n)) = 0, \end{aligned} \tag{40}$$

with

$$\alpha_i(n) = \frac{\phi_n^2(\lambda_i) - \gamma_i\psi_n^2(\lambda_i)}{\phi_n^1(\lambda_i) - \gamma_i\psi_n^1(\lambda_i)}, \qquad (i = 1, 2). \tag{41}$$

A direct calculation for equation (40) gives us

$$\begin{aligned} t_{11}(n) &= \frac{\alpha_1(n)\lambda_2 - \alpha_2(n)\lambda_1}{\lambda_1 - \lambda_2 + \alpha_2(n) - \alpha_1(n)}, & t_{12}(n) &= \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \alpha_2(n) - \alpha_1(n)}, \\ t_{21}(n) &= \frac{(\lambda_2 - \lambda_1)\alpha_1(n)\alpha_2(n)}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}, & t_{22}(n) &= \frac{\lambda_1\lambda_2(\alpha_1(n) - \alpha_2(n))}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}, \end{aligned} \tag{42}$$

where the parameters λ_i and γ_i ($\lambda_1 \neq \lambda_2, \gamma_1 \neq \gamma_2$) are suitably chosen such that all the denominators in (41), (42) are non-zero. Equations (1) and (41) present

$$\alpha_i(n+1) = \mu_i(n)/\nu_i(n), \qquad i = 1, 2, \tag{43}$$

with

$$\begin{cases} \mu_i(n) = \lambda_i r_n + \alpha_i(n)(\lambda_i + s_n), \\ \nu_i(n) = \alpha_i(n). \end{cases}$$

By using equations (43) and (42), we have

$$\begin{cases} t_{11}(n+1) = \frac{\mu_1(n)\nu_2(n)\lambda_2 - \mu_2(n)\nu_1(n)\lambda_1}{\nu_1(n)\nu_2(n)(\lambda_1-\lambda_2)+\mu_2(n)\nu_1(n)-\mu_1(n)\nu_2(n)}, \\ t_{12}(n+1) = \frac{(\lambda_2-\lambda_1)\nu_1(n)\nu_2(n)}{\nu_1(n)\nu_2(n)(\lambda_1-\lambda_2)+\mu_2(n)\nu_1(n)-\mu_1(n)\nu_2(n)}, \\ t_{21}(n+1) = \frac{(\lambda_2-\lambda_1)\mu_1(n)\mu_2(n)}{\lambda_1\mu_2(n)\nu_1(n)-\lambda_2\mu_1(n)\nu_2(n)}, \\ t_{22}(n+1) = \frac{\lambda_1\lambda_2(\mu_1(n)\nu_2(n)-\mu_2(n)\nu_1(n))}{\lambda_1\mu_2(n)\nu_1(n)-\lambda_2\mu_1(n)\nu_2(n)}. \end{cases} \tag{44}$$

Through direct but tedious calculations, from (42) and (44), we obtain the following equalities:

$$r_n t_{12}(n+1) - t_{21}(n) = 0, \qquad t_{11}(n+1) + s_n t_{12}(n+1) - t_{22}(n) = 0. \tag{45}$$

In fact, for the first equality of (45), from (44) and (43) we have

$$\begin{aligned} r_n t_{12}(n+1) &= \frac{r_n(\lambda_2 - \lambda_1)\nu_1(n)\nu_2(n)}{\nu_1(n)\nu_2(n)(\lambda_1 - \lambda_2) + \mu_2(n)\nu_1(n) - \mu_1(n)\nu_2(n)} \\ &= \frac{r_n(\lambda_2 - \lambda_1)\alpha_1(n)\alpha_2(n)}{\alpha_1(n)\alpha_2(n)(\lambda_1 - \lambda_2) + [\lambda_2 r_n + \alpha_2(n)(\lambda_2 + s_n)]\alpha_1(n) - [\lambda_1 r_n + \alpha_1(n)(\lambda_1 + s_n)]\alpha_2(n)} \\ &= \frac{(\lambda_2 - \lambda_1)\alpha_1(n)\alpha_2(n)}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)} \\ &= t_{21}(n). \end{aligned}$$

Similarly, we can find that the second equality of (45) is right.

Now, by virtue of equation (42), we obtain

$$\det T_n = (1 - t_{12}(n))(\lambda - \lambda_1)(\lambda - \lambda_2). \tag{46}$$

Hence, from all the above statements, we obtain following assertions.

Proposition 1. *The matrix $\widetilde{U}_n$ defined by (38) has the same form as U_n, that is*

$$\widetilde{U}_n = \begin{pmatrix} 0 & 1 \\ \lambda\widetilde{r}_n & \lambda + \widetilde{s}_n \end{pmatrix},$$

in which the transformation formulae between old and new potentials are defined by

$$\widetilde{r}_n = \frac{r_n - t_{21}(n)}{1 - t_{12}(n)}, \qquad \widetilde{s}_n = s_n + Dt_{22}(n) + t_{21}(n+1) - \frac{r_n - t_{21}(n)}{1 - t_{12}(n)} t_{12}(n). \tag{47}$$

The transformations (37) and (47): $(\varphi_n;\, p_n, q_n) \to (\widetilde{\varphi}_n; \widetilde{r}_n, \widetilde{s}_n)$ *is usually called a DT of eigenvalue problem (1). And equation (47) can be called a Bäcklund transformation (BT) between new and old potentials.*

Proof. Let $T_n^{-1} = T_n^*/\det T_n$ and

$$T_{n+1} U_n T_n^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}.$$

It is easy to verify that $\lambda f_{11}(\lambda, n)$, $\lambda f_{12}(\lambda, n)$, $f_{21}(\lambda, n)$ and $f_{22}(\lambda, n)$ are cubic-polynomials in λ, respectively. By virtue of (40) and (41), it can be verified that λ_i are roots of $f_{ij}(\lambda, n)$, $(i = 1, 2)$. Therefore, noting (46), we have

$$T_{n+1} U_n T_n^* = (\det T_n) P_n = (\det T_n) \begin{pmatrix} p_{11}^0 & p_{12}^0 \\ \lambda p_{21}^1 + p_{21}^0 & \lambda p_{22}^1 + p_{22}^0 \end{pmatrix},$$

where p_{ij}^l, $(i, j = 1, 2; l = 0, 1)$ are independent of λ. At the same time, the above equation can be rewritten as

$$T_{n+1} U_n = P_n T_n, \tag{48}$$

i.e.

$$\lambda t_{12}(n+1) r_n = [(1 - t_{12}(n))\lambda + t_{11}(n)] p_{11}^0 \lambda + \lambda t_{21}(n),$$
$$(1 - t_{12}(n+1))\lambda + t_{11}(n+1) + t_{12}(n+1)(\lambda + s_n) = p_{11}^0 t_{12}(n) + p_{12}^0 (\lambda + t_{22}(n)),$$
$$(\lambda + t_{22}(n+1))\lambda r_n = \left(\lambda p_{21}^1 + p_{21}^0\right)((1 - t_{12}(n))\lambda + t_{11}(n)) + \left(\lambda p_{22}^1 + p_{22}^0\right)(\lambda + t_{22}(n)),$$
$$\lambda t_{21}(n+1) + (\lambda + t_{22}(n+1))(\lambda + s_n) = \left(\lambda p_{21}^1 + p_{21}^0\right) t_{12}(n) + \left(\lambda p_{22}^1 + p_{22}^0\right)(\lambda + t_{22}(n)).$$

Equating the coefficients of $\lambda^i (i = 0, 1, 2)$ in above equations, noting equation (45), we have

$$p_{11}^0 = p_{12}^0 = 0, \qquad p_{12}^0 = p_{22}^1 = 1,$$
$$p_{21}^1 = \frac{r_n - t_{21}(n)}{1 - t_{12}(n)} = \widetilde{r}_n,$$
$$p_{22}^0 = s_n + Dt_{22}(n) + t_{21}(n+1) - \frac{r_n - t_{21}(n)}{1 - t_{12}(n)} t_{12}(n) = \widetilde{s}_n.$$

Thus we complete the proof. □

Proposition 2. *Under the transformation (47), the matrix* $\widetilde{V}_n^{\{+1\}}$ *defined by (38) has the same form as* $V_n^{\{+1\}}$*, that is*

$$\widetilde{V}_n^{\{+1\}} = \begin{pmatrix} -\frac{1}{2}\lambda - \widetilde{r}_{n-1} - \widetilde{s}_{n-1} & 1 \\ \lambda \widetilde{r}_n & \frac{1}{2}\lambda - \widetilde{r}_n \end{pmatrix}.$$

Proof. Let $T_n^{-1} = T_n^*/\det T_n$ and

$$\left(T_{nt} + T_n V_n^{\{+1\}}\right) T_n^* = \begin{pmatrix} g_{11}(\lambda, n) & g_{12}(\lambda, n) \\ g_{21}(\lambda, n) & g_{22}(\lambda, n) \end{pmatrix}.$$

Through direct calculation we know that $g_{11}(\lambda,n)$, $\lambda g_{12}(\lambda,n)$, $g_{21}(\lambda,n)$ and $g_{22}(\lambda,n)$ are cubic polynomials in λ, respectively. From (41) and (42), we have

$$\begin{aligned}&\lambda_i t_{12_t}(n)-t_{11_t}(n)-\alpha_i(n)t_{12_t}(n)-\alpha_{it}(n)t_{12}(n)=0,\\&t_{22_t}(n)+\alpha_i^{-1}(n)\lambda_i t_{21_t}(n)+\alpha_i^{-1}(n)\lambda_i\alpha_{it}(n)+\alpha_i^{-1}(n)\alpha_{it}(n)t_{22}(n)=0,\\&\alpha_{it}(n)=\lambda r_n+(\lambda_i+r_{n-1}+s_{n-1}+r_n)\alpha_i(n)-\alpha_i^2(n).\end{aligned}\tag{49}$$

Equations (40) and (49) tell us that λ_i $(i=1,2)$ are roots of $g_{ij}(\lambda,n)$, $(i,j=1,2)$. Therefore, noting (46), we have

$$\left(T_{nt}+T_nV_n^{\{1\}}\right)T_n^*=(\det T_n)R_n=(\det T_n)\begin{pmatrix}r_{11}^1\lambda+r_{11}^0 & r_{12}^0\\ r_{21}^0 & r_{22}\lambda+r_{22}^0\end{pmatrix},$$

that is

$$T_{nt}+T_nV_n^{\{1\}}=R_nT_n,\tag{50}$$

where $r_{ij}^l(i,j=1,2;l=0,1)$ are independent of λ. Comparing the coefficients of $\lambda^i(i=0,1,2)$ in (50), noting (45), we arrive at

$$\begin{aligned}&r_{11}^1=-r_{22}^1=-\frac{1}{2},\qquad r_{12}^0=1,\qquad r_{21}^0=0,\\&r_{21}^1=\frac{r_n-t_{21}(n)}{1-t_{12}(n)}=\widetilde{r}_n=-r_{22}^0,\\&r_{11}^0=-\frac{r_{n-1}-t_{21}(n-1)}{1-t_{12}(n-1)}-s_{n-1}-Dt_{22}(n-1)-t_{21}(n)+\frac{r_{n-1}-t_{21}(n-1)}{1-t_{12}(n-1)}t_{12}(n-1)\\&\quad=-\widetilde{r}_{n-1}-\widetilde{s}_{n-1}.\end{aligned}$$

The proof is thus completed. □

Since the fact of equivalence between differential-difference equation (4*a*) and the discrete zero-curvature equation $U_{nt}-V_{n+1}^{\{+1\}}U_n+U_nV_n^{\{+1\}}=0$, from propositions 1 and 2, we obtain the following theorem.

Theorem. *Under the transformation*

$$\widetilde{r}_n=\frac{r_n-t_{21}(n)}{1-t_{12}(n)},\qquad \widetilde{s}_n=s_n+Dt_{22}(n)+t_{21}(n+1)-\frac{r_n-t_{21}(n)}{1-t_{12}(n)}t_{12}(n).$$

If r_n, s_n solve the positive relativistic Toda type lattice equation (4a), then so do $(\widetilde{r}_n,\widetilde{s}_n)$.

As below, an exact solution of the positive relativistic Toda type lattice equation (4*a*) will be given by using DT. Substituting the trivial solution $r_n=s_n=1$ of (4*a*) into (1) and (12), we then have

$$\varphi_{n+1}=\begin{pmatrix}0 & 1\\ \lambda & \lambda+1\end{pmatrix}\varphi_n,\qquad \varphi_{nt}=\begin{pmatrix}-\frac{1}{2}\lambda-2 & 1\\ \lambda & \frac{1}{2}\lambda-1\end{pmatrix}\varphi_n,$$

from which two real basic solutions of equation (4*a*) are presented as

$$\phi_n=\left(\frac{\lambda+1+\sqrt{(\lambda+1)^2+4\lambda}}{2}\right)^n\exp\left(\frac{-3+\sqrt{(\lambda+1)^2+4\lambda}}{2}t\right)\begin{pmatrix}2\\ \lambda+1+\sqrt{(\lambda+1)^2+4\lambda}\end{pmatrix},$$

$$\psi_n=\left(\frac{\lambda+1-\sqrt{(\lambda+1)^2+4\lambda}}{2}\right)^n\exp\left(\frac{-3-\sqrt{(\lambda+1)^2+4\lambda}}{2}t\right)\begin{pmatrix}2\\ \lambda+1-\sqrt{(\lambda+1)^2+4\lambda}\end{pmatrix},$$

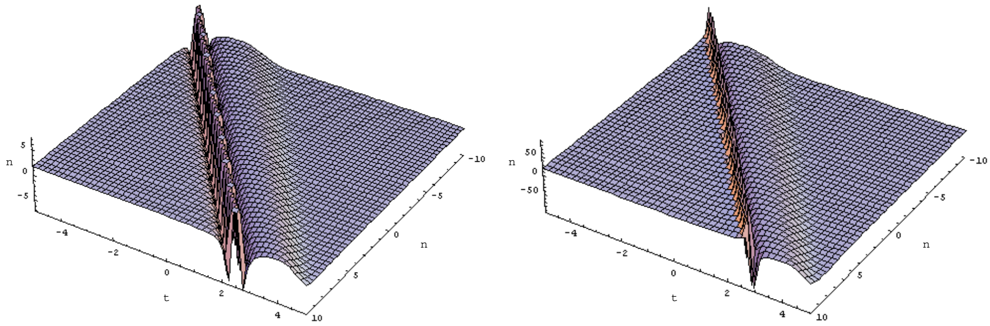

Figure 1. One-soliton solutions with $\lambda_1 = -6.5, \lambda_2 = -6.4, \gamma_1 = 0.6, \gamma_2 = 0.5$.

where $\lambda \in \mathbb{R} - \{0\}$. From (41) we have

$$\alpha_i(n) = \frac{\xi_i^{2n} \exp(t\sqrt{(\lambda_i+1)^2+4\lambda_i})(\lambda_i - 1 + \sqrt{(\lambda_i+1)^2+4\lambda_i}) - \gamma_i(\lambda_i - 1 - \sqrt{(\lambda_i+1)^2+4\lambda_i})}{2\xi_i^{2n}\exp(t\sqrt{(\lambda_i+1)^2+4\lambda_i}) - 2\gamma_i},$$

with $\xi_i = \frac{\lambda_i+1+\sqrt{(\lambda_i+1)^2+4\lambda_i}}{2\mathrm{i}\sqrt{|\lambda_i|}}$, $(i = 1, 2)$ where $\mathrm{i}^2 = -1$. So the new solutions of relativistic Toda type lattice equation (4*a*) by applying (47) can be shown as

$$\widetilde{r}_n = 1 + \frac{\lambda_1 - \lambda_2}{\alpha_2(n) - \alpha_1(n)} - \frac{\alpha_1(n)\alpha_2(n)(\lambda_2 - \lambda_1)}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}\left[1 - \frac{\lambda_2 - \lambda_1}{\alpha_2(n) - \alpha_1(n)}\right],$$

$$\widetilde{s}_n = 1 + 2\frac{\lambda_1\lambda_2[\alpha_1(n)\alpha_2(n)(\lambda_1 - \lambda_2) + \alpha_2(n)\lambda_1 - \alpha_1(n)\lambda_2]}{\lambda_1\lambda_2(\alpha_1(n) - \alpha_2(n)) + \alpha_1(n)\alpha_2(n)(\lambda_1 - \lambda_2)} - \left[1 - \frac{\alpha_1(n)\alpha_2(n)(\lambda_2 - \lambda_1)}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}\right]\frac{\lambda_1 - \lambda_2}{\alpha_2(n) - \alpha_1(n)} - \frac{\lambda_1\lambda_2(\alpha_1(n) - \alpha_2(n))}{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}.$$

The plots of solutions $\widetilde{r}, \widetilde{s}$, for equation (4*a*) by using DT (23) and (33) are given in figure 1 with the parameters chosen as $\lambda_1 = -6.5, \lambda_2 = -6.4, \gamma_1 = 0.6, \gamma_2 = 0.5$. It is easy to verify that the solutions of equation (4*a*) by DT are one-soliton solutions. Furthermore, if the resulting solutions are taken as the new starting point, we can make the DT once again and engender another set of new explicit solutions. This process can be done continuously and the multi-soliton solutions result usually.

5. Summary and remarks

In this paper, based on a discrete isospectral problem, two hierarchies of nonlinear integrable lattice equations are derived. It is shown that every equation in the resulting models is integrable in Liouville sense. It is also shown that these two hierarchies correspond to positive and negative power expansions concerning spectral parameter, respectively. The typical system (4*a*) of the positive hierarchy (11) is relativistic Toda type lattice. In addition, with the help of the gauge transformations of Lax pairs, a Darboux transformation is established for the first nonlinear equations of resulting hierarchies, from which the soliton solutions result.

Searching for new integrable discrete systems is still a significant but difficult task in soliton theory. Motivated by [10], the procedure for new lattice systems is provided again in this study. We believe that the present study can be used for other applications even for more complicated spectral problems in higher order. Moreover, we should emphasize that the mathematical and physical background as well as the deeper properties, such as the symmetries, infinitely many conservation laws, nonlinearization and so on of the relativistic Toda type lattice hierarchy (11) would be fulfilled in other papers.

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